Completely reducible subcomplexes of spherical buildings

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April 15 2008

Abstract

A completely reducible subcomplex of a spherical building is a spherical building.

By a *sphere* we mean metric space isometric to the unit sphere $\mathbb{S}^m \subseteq \mathbb{R}^{m+1}$, endowed with the spherical metric d. The distance of $u, v \in \mathbb{S}^m \subseteq \mathbb{R}^{m+1}$ is given by $\cos(d(u,v)) = u \cdot v$ (standard inner product). Recall that a geodesic metric space Z is CAT(1) if no geodesic triangle of perimeter $< 2\pi$ in Z is thicker than its comparison triangle in the 2-sphere $\mathbb{S}^2$. It follows that any two points at distance $< \pi$ can be joined by a unique geodesic segment (an isometric copy of a closed interval). A subset Y of a CAT(1)-space Z is *convex* if the following holds: for all $x, y \in Y$ with $d(x,y) < \pi$, the geodesic segment $[x,y]$ is contained in Y. It is clear from the definition that arbitrary intersections of convex subsets are convex (and CAT(1)).

1 Convex sets in Coxeter complexes

Let Σ be an n-dimensional spherical Coxeter complex, let $\bar{\Sigma}$ be a simplicial complex which refines the triangulation of Σ and which is invariant under the Coxeter group W and $\pm id$. Examples of such triangulations are Σ itself and its barycentric subdivisions. In the geometric realization, the simplices are assumed to be spherical. The *span* of a subset of a sphere is the smallest subsphere containing the set.

We assume now that $A \subseteq \bar{\Sigma}$ is an m-dimensional subcomplex whose geometric realization $|A|$ is convex.

1.1 Lemma *Let $a \in A$ be an m-simplex. Then $|A| \subseteq \operatorname{span}|a|$.*

Proof. Assume this is false. Let $u \in |A| \setminus \operatorname{span}|a|$. Then $-u \notin |a|$ and $Y = \bigcup\{[u,v] \mid v \in |a|\}$ is contained in $|A|$. But Y is a cone over $|a|$ and in particular $m+1$-dimensional, a contradiction. □

We choose an m-simplex $a \in A$ and put

$$S = \operatorname{span}|a| \cap |\bar{\Sigma}|;$$

this is an m-sphere containing $|A|$. Recall that an m-dimensional simplicial complex is called *pure* if every simplex is contained in some m-simplex.

1.2 Lemma *A is pure.*

Proof. It suffices to consider the case $m \geq 1$. Let $a \in A$ be an m-simplex, and assume that $b \in A$ is a lonely simplex of maximal dimension $\ell < m$. Then $int(-b)$ is disjoint from $int(a)$. Let v be an interior point of a and u an interior point of b and consider the geodesic segment $[u, v] \subseteq |A|$. If b is a point, the existence of the geodesic shows that b is contained in some higher dimensional simplex, a contradiction. If $\ell \geq 1$, then $[u, v]$ intersects $int(b)$ in more than two points (because b is lonely), so v is in the span of b. This contradicts $dim(a) > dim(b)$. □

1.3 Lemma *If there exists an m-simplex $a \in A$ with $-a \in A$, then $|A| = S$.*

Proof. Then any point in S lies on some geodesic of length $< \pi$ joining a point in $|a|$ with a point in $|-a|$. □

Topologically, the convex set $|A|$ is either an m-sphere or homeomorphic to a closed m-ball. For $m \geq 2$, these spaces are strongly connected (i.e. they cannot be separated by $m-2$-dimensional subcomplexes [1]). It follows that A is a chamber complex, i.e. the chamber graph $C(A)$ (whose vertices are the m-simplices and whose edges are the $m-1$ simplices) is connected [1]. If $m = 1$, then $|A|$ is a connected graph and hence strongly connected.

1.4 Lemma *If $m \geq 1$, then A is a chamber complex.* □

2 Results by Balser-Lytchak and Serre

We now assume that X is a simplicial spherical building modeled on the Coxeter complex Σ. By means of the coordinate charts for the apartments we obtain a metric simplicial complex $\bar{X}$ refining X, which is modeled locally on $\bar{\Sigma}$. In this refined complex $\bar{X}$, we call two simplices a, b *opposite* if $a = -b$ in some (whence any) apartment containing both. We let $opp(a)$ denote the collection of all simplices in $\bar{X}$ opposite a. The geometric realization $|\bar{X}|$ is CAT(1). Furthermore, any geodesic arc is contained in some apartment.

We assume that $A \subseteq \bar{X}$ is an m-dimensional subcomplex and that $|A|$ is convex. For any two simplices $a, b \in A$, we can find an apartment $\bar{\Sigma}$ containing a and b. The intersection $|A| \cap |\bar{\Sigma}|$ is then convex, so we may apply the results of the previous section to it. We note also that $|A|$ is CAT(1).

2.1 Lemma *A is a pure chamber complex.*

Proof. Let $a \in A$ be an m-simplex and let $b \in A$ be any simplex. Let $\bar{\Sigma}$ be an apartment containing a and b. Since $|\bar{\Sigma}| \cap |A|$ is m-dimensional and convex, we find an m-simplex $c \in A \cap \bar{\Sigma}$ containing b. Similarly we see that A is a chamber complex.

The next results are due to Serre [9] and Balser-Lytchak [2, 3].

2.2 Lemma *If there is a simplex $a \in A$ with $opp(a) \cap A = \varnothing$, then $|A|$ is contractible.*

Proof. We choose u in the interior of a. Then $d(u, v) < \pi$ for all $v \in |A|$, so $|A|$ can be contracted to u along these unique geodesics. □

2.3 Proposition *If there is an m-simplex a in A with $opp(a) \cap A \neq \varnothing$, then every simplex $a \in A$ has an opposite in A.*

Proof. Let $a, b \in A$ be opposite m-simplices, let $\bar{\Sigma}$ be an apartment containing both and let $S \subseteq |\bar{\Sigma}|$ denote the sphere spanned by a, b. Then $S \subseteq |A|$. Let c be any m-simplex in A. If c is not opposite a, we find interior points u, v of c, a with $d(u, v) < \pi$. The geodesic arc $[u, v]$ has a unique extension in S. Along this extension, let w be the point with $d(u, w) = \pi$ and let c' be the smallest simplex containing w. Then c' is opposite c.

Thus every m-simplex in A has an opposite, and therefore every simplex in A has an opposite. □

In this situation where every simplex has an opposite, A is called A *completely reducible.* If every simplex of a fixed dimension $k \leq m$ has an opposite in A, then clearly every vertex in A has an opposite. Serre [9] observed that the latter already characterizes complete reducibility.

2.4 Proposition *If every vertex in A has an opposite, then A is completely reducible.*

Proof. We show inductively that A contains a pair of opposite k-simplices, for $0 \leq k \leq m$. This holds for $k = 0$ by assumption, and we are done if $k = m$ by 2.3. So we assume that $0 \leq k < m$.

Let a, a' be opposite k-simplices in A and let $b \in A$ be a vertex which generates together with a a $k + 1$-simplex (recall that A is pure, so such a vertex exists). We fix an apartment $\bar{\Sigma}$ containing a, b and a'. The geodesic convex closure Y of b and $|a| \cup |a'|$ in the sphere $|\bar{\Sigma}|$ is a $k + 1$-dimensional hemisphere (and is contained in $|A|$). Let $b' \in A$ be a vertex opposite b. A small ε-ball in Y about b generates together with b' a $k + 1$-sphere $S \subseteq |A|$. Because $\dim S = k + 1$, there exists a point $u \in S$ such that the minimal simplex c containing u has dimension at least $k+1$. Let u' be the opposite of u in S, and c' the minimal simplex containing u'. Then c, c' is a pair of opposite simplices in A of dimensions at least $k + 1$. □

3 Completely reducible subcomplexes are buildings

We assume that A is m-dimensional, convex and completely reducible. If $m = 0$, then A consists of a set of vertices which have pairwise distance π. This set is, trivially, a 0-dimensional spherical building. So we assume now that $1 \leq m \leq n$. Two opposite m-simplices $a, b \in A$ determine an m-sphere $S(a, b)$ which we call a *Levi sphere.*

3.1 Lemma *If $a, b \in A$ are m-simplices, then there is a Levi sphere containing a and b.*

Proof. This is true if b is opposite a. If b is not opposite a, we choose interior points $u \in int(a)$ and $v \in int(b)$, and a simplex $c \in A$ opposite b. The geodesic $[u, v]$ has a unique continuation $[v, w]$ in the Levi sphere $S(b, c)$, such that $d(u, w) = \pi$. Let $\bar{\Sigma}$ be an apartment containing the geodesic arc $[u, v] \cup [v, w]$ and let d be the smallest simplex in $\bar{\Sigma}$ containing w. Then d is in A and opposite a, so the there is a Levi sphere $S(a, d)$ containing $[u, v] \cup [v, w]$. Since b is the smallest simplex containing v, it follows that $b \in S(a, d)$. □

Since A is pure, we have the following consequence.

3.2 Corollary *Any two simplices $a, b \in A$ are in some Levi sphere.* □

We call an $m-1$-simplex $b \in A$ *singular* if it is contained in three different m-simplices. The following idea is taken from Caprace [5]. Two m-simplices are *t-equivalent* if there is a path between them in the dual graph which never crosses a singular $m-1$-simplex. The t-class of a is contained in all Levi spheres containing a.

3.3 Lemma *Let b be a singular $m-1$-simplex. Let S be a Levi sphere containing b and let $H \subseteq S$ denote the great $m-1$-sphere spanned by $|b|$. Then H is the union of singular $m-1$-simplices.*

Proof. Let a be an m-simplex containing b which is not in S and let $-b$ denote the opposite of b in S. Let S' be a Levi sphere containing a and $-b$ and consider the convex hull Y of $|a| \cup |-b|$ in S'. Then Y is an m-hemisphere. The intersection $Y \cap S$ is convex, contains the great sphere H, and is different from Y, so $Y \cap S = H$. □

We call H a *singular great sphere.* Along singular great spheres, we can do 'surgery':

3.4 Lemma *Let S, H, Y be as in the previous lemma. Let $Z \subseteq S$ be a hemisphere with boundary H. Then $Z \cup Y$ is a Levi sphere.*

Proof. We use the same notation as in the previous lemma. Let $c \subseteq Z$ be an m-simplex containing $-b$, then $|c| \cup H$ generates Z. Let S' be a Levi sphere containing c and a. Then $Z \cup Y \subseteq S'$ and $Z \cup Y = H$, whence $S' = Z \cup Y$. □

3.5 Lemma *Let S be a Levi sphere and let $H, H' \subseteq S$ be singular great spheres. Let s denote the metric reflection of S along H. Then $s(H')$ is again a singular great sphere.*

Proof. We use the notation of the previous lemma. Let b' be a singular $m-1$-simplex in $H' \cap Z$. Let $-b'$ denote its opposite in the Levi sphere $S' = Z \cup Y$. We note that the interior of b is disjoint from S. Let b'' be the opposite of $-b'$ in the Levi sphere $S'' = (S \setminus Z) \cup Y$. Then b'' is a singular $m-1$-simplex in S, and b'' is precisely the reflection $s(b')$ of b' along H. □

For every Levi sphere S we obtain in this way a finite reflection group W_S which permutes the singular great spheres in S. As a reprentation sphere, S may split off a trivial factor S_0, the intersection of all singular great spheres in S. We let S_+ denote its orthogonal complement, $S = S_0 * S_+$. The intersections of the singular great spheres with S_+ turn S_+ into a spherical Coxeter complex, with Coxeter group W_S. Let $F \subseteq S$ be a fundamental domain for W_S, i.e. $F = C * S_0$, where $C \subseteq S_+$ is a Weyl chamber. The geometric realization of the t-class of any m-simplex in F is precisely F.

3.6 Lemma *If two Levi spheres S, S' have an m-simplex a in common, then there is a unique isometry $\varphi : S \longrightarrow S'$ fixing $S \cap S'$ pointwise. The isometry fixes S_0 and maps W_S isomorphically onto $W_{S'}$.*

Proof. The intersection $Y = S \cap S'$ contains the fundamental domain F. Since F is relatively open in S, there is a unique isometry $\varphi : S \longrightarrow S'$ fixing Y. The Coxeter group W_s is generated by the reflections along the singular $m-1$-simplices in Y. Therefore φ conjugates W_S onto $W_{S'}$. Finally, Y contains S_0. □

3.7 Corollary *If two Levi spheres S, S' have a point u in common, then $S_0 = S'_0$. Furtheremore, there exists an isometry $\varphi : S \longrightarrow S'$ which fixes $S \cap S'$ and which conjugates W_S to $W_{S'}$.*

Proof. Let a, a' be m-simplices in S and S' containing u, and let S'' be a Levi sphere containing a and a'. We compose $S \longrightarrow S'' \longrightarrow S'$.

3.8 Theorem *Let A be completely reducible. Then there is a thick spherical building Z such that $|A|$ is the metric realization of $Z * \mathbb{S}^0 * \cdots * \mathbb{S}^0$.*

Proof. Let S be a Levi sphere and let $k = \dim S_0 + 1$. We make S_0 into a Coxeter complex with Coxeter group $W_0 = \mathbb{Z}/2^k$ (we fix an action, this is not canonical). By the previous Corollary, we can transport the simplicial structure on S unambiguously to any Levi sphere in A. □

For $A = X$, this is Scharlau's reduction theorem for weak spherical buildings [8] [5].

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